THE DOUBLE CENTRALIZER THEOREM (DCT)

The purpose of this worksheet is to work through the details of the proof of the Double Centralizer Theorem (DCT); in addition, we will discuss some consequences of the DCT to representation theory and play around with some concrete examples.

1. A brief review of the Artin-Wedderburn Theorem (AWT)

This section's sole purpose is to serve as a review of the Artin-Wedderburn Theorem (AWT) from the Artin-Wedderburn worksheet:

Theorem 1.1 (Artin-Wedderburn Theorem (general version))**.** *Let R be a left semisimple ring. Then for some $m \geqslant 0$, positive integers $n_1, \ldots, n_m$, and division rings $D_1, \ldots, D_m$, there is a ring isomorphism*

$$R \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_m}(D_m).$$

Moreover,

(1) m is the number of isomorphism classes of simple left R-modules.
(2) Say $M_1, \ldots, M_m$ are simple modules forming a complete set of representatives of these isomorphism classes. Then, after reordering, $D_i \cong \mathrm{End}_R(M_i)^{\mathrm{op}}$ and
(3) n_j is the number of times summands isomorphic to M_j occur in the decomposition of R into a direct sum of simple left modules.

Moreover, the data $(m; n_1, \ldots, n_m; D_1, \ldots, D_m)$ is unique up to a permutation of $\{1, \ldots, m\}$ and isomorphisms of division rings.

Remark 1.2.
The AWT tells us that left (and right) semisimple algebras have a very nice structure based on matrix rings over division rings. In particular, these semisimple algebras are, in a sense, the next best thing after vector spaces over a field. This is remarkable as rings, even commutative ones, tend to evade structure theorems without moderately strong hypothesis (e.g. the Cohen Structure Theorem for commutative, Noetherian, complete local rings).

2. The DCT and its proof

Definition 2.1. For a G-module W and simple G-module U, the *isotypic component of W of type U* is the sum of all submodule of W isomorphic to U. The isotypic components form a direct sum which is all of W if and only if W is semisimple. In that case it is call the *isotypic decomposition.*

Theorem 2.1. *Let W be a finite dimensional vector space over a field K. Let $A \subset \mathrm{End}_K(W)$ be a semisimple (left and right semisimple) K-subalgebra. Let*

$$A' = \{b \in \mathrm{End}_K(W) \mid ab = ba \text{ for all } a \in A\}$$

be the centralizer of A in $\mathrm{End}_K(W)$. Then:

(1) A' is semisimple and $(A')' = A$.

(2) *W has a unique decomposition $W = W_1 \oplus \cdots \oplus W_r$ into simple, non-isomorphic $A \otimes A'$-modules W_i. In addition, this is the isotypic decomposition of W as an A-module and as an A'-module.*
(3) *Each simple factor W_i is of the form $U_i \otimes_{D_i} U_i'$, where U_i is a simple A-module, U_i' a simple A'-module, and D_i is the division algebra $\mathrm{End}_A(U_i)^{\mathrm{op}} = \mathrm{End}_{A'}(U_i')^{\mathrm{op}}$.*

Proof. By AWT, consider the isotypic decomposition of A as a semisimple K-subalgebra of $\mathrm{End}_K(W)$:

$$A = \prod_{i=1}^{r} A_i,$$

where $A_i \cong \mathrm{Mat}_{n_i}(D_i)$ with a division algebra $D_i \supset K$. Since $A \subseteq \mathrm{End}_K(W)$, then W is a A-module.

Exercise 1.
Prove that if A is left (right) semisimple, then every left (right) module over A is left (right) semisimple.

By Exercise 1, we have that W is a semisimple A-module. Therefore, we may write

$$W \cong W_1 \oplus \cdots \oplus W_r$$

where $W_i \cong U_i^{s_i}$ with U_i a simple A_i-module (it is also a simple A-module).

Exercise 2.
(1) *Let $\mathrm{Mat}_n(D)$ be a matrix ring over a division ring D. Show that any nontrivial two sided ideal of $\mathrm{Mat}_n(D)$ is isomorphic to the space of column vectors of elements in D of size n.*
(2) *Conclude that $U_i \cong D_i^{n_i}$, where we view $D_i^{n_i}$ as a A-module via the composition $A \xrightarrow{pr} A_i \cong \mathrm{Mat}_{n_i}(D_i)$.*

We now analyze A' and its relationship to A.

Exercise 3.
(1) *Show that $A' = \mathrm{End}_A(W)$ and $A' = \prod \mathrm{End}_A(W_i)$.*
(2) *Set $A_i' = \mathrm{End}_A(W_i)$. Show that $A_i' \cong M_{s_i}(D_i')$, where $D_i' = \mathrm{End}_{A_i}(U_i) = D_i^{\mathrm{op}}$.*
(3) *Show that $\dim_K(A_i \otimes_K A_i') = \dim_K(\mathrm{End}(W_i))$.*

Exercise 4. *Proof of Part (1):*
Conclude $(A')' = A$ by showing $\dim_K(A) = \dim_K((A')')$ using the same reasoning as in Exercise 3 on A' and $(A')'$.

Exercise 5. *Proof of Part (2):*
In the text, the authors call upon a canonical algbra homomorphism $A_i \otimes A_i' \to \mathrm{End}(W_i)$.
(1) *Describe the canonical algebra homomorphism*

$$\mathrm{Mat}_{n_i}(D_i) \otimes_K \mathrm{Mat}_{s_i}(D_i^{\mathrm{op}}) \to \mathrm{End}_K(W_i).$$

(2) *Show that $A_i \otimes_K A_i'$ a simple K-algebra.*

(3) *Conclude that the canonical algebra homomorphism from Exercise 5 part (1) is an isomorphism.*
(4) *Use Schur's Lemma to show that W_i is a simple $A \otimes_K A'$-module.*

Exercise 6. *Proof of Part (3):*

(1) *For any ring R, consider R as a right R-module.*
 (a) *Show that R^{op} as a left R-module via $r \cdot s = rs$ for all $r \in R$ and $s \in R^{\mathrm{op}}$.*
 (b) *One might be tempted to describe R^{op} as a left R-module via the action $r \cdot s = sr$. If R is not commutative, convince yourself that this need not give R^{op} a left R-module structure.*
 (c) *Show that $R \otimes_R R^{\mathrm{op}} \cong R$.*
 (d) *For any natural numbers n and m, show that $R^n \otimes_R (R^{\mathrm{op}})^m \cong R^{nm}$.*
(2) *Recall that $U_i \cong D_i^{n_i}$. Consider U_i as a right D_i-module and $U_i' = (D_i^{\mathrm{op}})^{s_i}$ as a left D_i-module.*
 (a) *Show that $U_i \otimes_{D_i} U_i'$ is an $A \otimes A'$ module in a canonical way.*
 (b) *Show that $U_i \otimes_{D_i} U_i' \cong U_i^{s_i} \cong W_i$ as $A \otimes A'$-modules.*
 (c) *Conclude part (3) of the DCT.*

□

3. A Consequence of the DCT in Representation Theory

First, we describe the setting we will be considering in this section. Let K be any field and V a finite dimensional K-vector space. Define the n-fold tensor product of V by

$$V^{\otimes n} := V \underbrace{\otimes \cdots \otimes}_{n-times} V.$$

Given an element $g \in \mathrm{GL}_K(V)$, define

$$g \cdot (v_1 \otimes \cdots \otimes v_n) := g(v_1) \otimes \cdots \otimes g(v_n);$$

this gives rise to a linear action of $\mathrm{GL}(V)$ on $V^{\otimes m}$. On the other hand, the symmetric group S_n on n-letters, acts linearly on $V^{\otimes m}$, as well by defining

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma n}.$$

We denote by $\langle \mathrm{GL}(V) \rangle$ and $\langle S_n \rangle$ the subalgebras generated by the image of $\mathrm{GL}(V)$ and S_n in $\mathrm{End}_K(V^{\otimes m})$, respectively.

Exercise 7. *Show that the two actions above commute with each other in* $\mathrm{End}_K(V^{\otimes m})$.

Theorem 3.1. *For any field K, we have* $\mathrm{End}_{S_n}(V^{\otimes n}) = \langle \mathrm{GL}(V) \rangle$. *Moreover, if* $\mathrm{char}(K)$ *does not divide $n!$, then* $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes}) = \langle S_n \rangle$.

Proof. The key is the following exercise

Exercise 8. *Let W be a finite dimensional vector space and $X \subset W$ a Zariski-dense subset of W. Then the linear span of tensors $x \underbrace{\otimes \cdots \otimes}_{n-times} x$ with $x \in X$ is the subspace $\Sigma_n \subset W^{\otimes n}$ of all symmetric tensors (i.e all tensors $v_1 \otimes \cdots \otimes v_n \in V^{\otimes m}$ such that $\sigma(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$ for all $\sigma \in S_n$).*

Exercise 9. *Consider V as in the statement of Theorem 2.2.*

(1) *The map*

$$\gamma : \mathrm{End}(V)^{\otimes n} \to \mathrm{End}(V^{\otimes n})$$

defined by setting

$$\gamma(A_1 \otimes \cdots \otimes A_n)(v_1 \otimes \cdots \otimes v_m) = Av_1 \otimes \cdots \otimes A_n v_n$$

is an isomorphism of K-algebras.

(2) *Show that the map γ from part (1) induces an isomorphism between the symmetric tensors in $\mathrm{End}_K(V)^{\otimes n}$ and the subalgebra $\mathrm{End}_{S_n}(V^{\otimes n})$ of $\mathrm{End}_K(V^{\otimes n})$.*

(3) *Use Exercise 8 with $X := \mathrm{GL}(V) \subseteq W := \mathrm{End}_K(V)$ to conclude part 1 of the theorem.*

(4) *Use the DCT to prove part 2 of the theorem.*

□

Theorem 3.2. *Assume that $\mathrm{char}(K)$ does not divide $n!$.*

(1) *The two subalgebras $\langle S_n \rangle$ and $\langle \mathrm{GL}(V) \rangle$ of $\mathrm{End}_K(V^{\otimes n})$ are semisimple and are centralizers of each other.*

(2) *There is a canonical decomposition of $V^{\otimes n}$ as an $S_n \times \mathrm{GL}(V)$-module into simple non-isomorphic $S_n \times \mathrm{GL}(V)$-modules V_λ:*

$$V^{\otimes n} = \bigoplus \lambda V_\lambda.$$

(3) *Each simple factor V_λ is of the form $M_\lambda \otimes L_\lambda$, where M_λ is a simple S_m-module and L_λ is a simple $\mathrm{GL}(V)$-module. Moreover, the modules M_λ (respectively, L_λ) are all non-isomorphic.*

Proof.

Exercise 10.

(1) *Prove part (1).*

(2) *Prove part (2).*

(3) *Prove part (3) in the case when K is algebraically closed. The general case will be proven later in the text.*

□

Exercise 11. *Assume the setup of Theorem 3.2. For those of us with some background in category theory, show that the assignment*

$$L_\lambda(-) : \mathbf{K-vect} \to \mathbf{GL(V)-Rep}$$

defined by $L_\lambda(V) = L_\lambda(V)$ as in Theorem 3.2, is a functor from the category of finite dimensional K-vector spaces to the category of $GL(V)$-representations. This functor $L_\lambda(-)$ is usually called the Schur functor or the Weyl functor

The following are exercises in the exercise sheet from 10/29/24:

Exercise 12. *Show that the isotypic componnet in $V^{\otimes n}$ of the trivial representation of S_n is the symmetric power $S^n(V)$.*

Exercise 13.
If $\dim(V) \geqslant n$, show that every irreducible representation of S_m occurs in $V^{\otimes n}$.

The following are new exercises:

Exercise 14. *Let* $V = \mathbb{C}^3$*.*
Decompose $V^{\otimes 3}$ *as in Theorem 3.2. In particular, find each* M_λ *and* L_λ*.*
Decompose $V^{\otimes 4}$ *as in Theorem 3.2. In particular, find each* M_λ *and* L_λ*.*
Decompose $V^{\otimes 2}$ *as in Theorem 3.2. In particular, find each* M_λ *and* L_λ*. Does the regular represenation of* S_3 *appear as an* M_λ*?*

Exercise 15. *Let* $V = \mathbb{F}_2^2$*.*

1. *Is* $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes 2}) = \langle S_2 \rangle$*?*
2. *Can we write* $V^{\otimes 2}$ *as a direct summand of irreducible* S_2 *representations as in the AWT?*