SOME FOUNDATIONS OF REPRESENTATION THEORY

The purpose of these notes is to introduce a few notions and concepts that are utilized in representation theory. Many thanks go out to Jack Jeffries for allowing me to take bits and pieces from his TeX file for MATH 901 Notes.

1. What is a Representation

Definition 1.1. Let R be a commutative ring. A representation of G over R is an R-module M together with a group homomorphism $\rho : G \to \operatorname{Aut}_R(M)$. We call (M, ρ) an R-linear G-representation.

One often simply says that M is a representation of G if the homomorphism ρ is understood. Often we will only be concerned about representations of a group over a field K, but for the time being, we will not limit ourselves.

Exercise 1.
(1) *Let R be a commutative ring and (M, ρ) a G-representation over R.*

Show that the map

$$G \times M \longrightarrow M$$
$$(g, m) \longrightarrow g \cdot m := \rho(g)(m)$$

satisfies the properties
  (a) $e \cdot m = m$
  (b) $gh \cdot m = g \cdot (h \cdot m)$
  (c) $g \cdot (m + n) = (g \cdot m) + (g \cdot n)$
  (d) $g \cdot rm = r(g \cdot m)$.

In particular, the first two conditions say that G acts on M in the sense of group action on a set, and the last two say that the action of any element is by an R-linear map.

(2) *Conversely, show that any such function $\psi : G \times M \to M$ satisfying the properties above yields a R-linear representation (M, ρ) of G.*

Example 1.2.
(1) For any group G, and any R-module V, there is the *trivial representation* $\rho : G \to \operatorname{Aut}_R(V)$ where $\rho(g) = 1_V$ for all $g \in G$. In this action, every element acts trivially on M.
(2) Any representation on $V = R$ is determined by specifying a group homomorphism $\rho : G \to \operatorname{Aut}_R(R) \cong R^{\times}$.
(3) If $G = C_n = \langle g \rangle$ (the multiplicative cyclic group of order n) and $R = \mathbb{C}$, there are n possible such homomorphisms, determined by $\rho(g) = e^{\frac{2\pi k i}{n}}$ where $0 \leqslant k \leqslant n - 1$.
(4) The *sign representation* of the symmetric group S_n, given by the group homomorphism which assigns to each permutation its sign, regarded as an element of the arbitrary ring R.
(5) The symmetric group S_n acts on a free R-module with basis $b_1, \ldots, b_n$ by permuting coordinates: $\rho(\sigma)(b_i) = b_{\sigma(i)}$. For a concrete example, S_3 acts on $\mathbb{R}^3$, where, for example $(132) \cdot (a_1, a_2, a_3) = (a_2, a_3, a_1)$.

(6) Let $G = D_{2n}$, symmetries of the equilateral polygon on n vertices. Then G acts linearly on $V = \mathbb{R}^2$ by rotations and reflections. If G is generated by r (rotation by $2\pi/n$) and l (reflection about the y-axis), then the associated group homomorphism $\rho : G \to \mathrm{GL}_2(\mathbb{R})$ maps

$$\rho(r) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \qquad \rho(l) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(7) Let $R = K$ be a field,$V = K^2$, and let $G = (K, +)$. We see that the assignment

$$\rho : G \to \mathrm{GL}_2(K) \quad \rho(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

is a representation. In particular, if $K = \mathbb{F}_p$, this is a representation of C_p.

Definition 1.3. If $\rho : G \to \mathrm{Aut}_R(V)$ and $\omega : G \to \mathrm{Aut}_R(W)$ are R-linear representations of G on V and W respectively then a *G-equivariant map* from V to W is an R-module homomorphism $f : V \to W$ such that $f(gv) = gf(v)$ for all $v \in V$. Equivalently the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \phi(g)}\downarrow & & \downarrow{\scriptstyle \psi(g)} \\ V & \xrightarrow{f} & W \end{array}$$

Exercise 2. *Let $R = \mathbb{C}$. Fix $V = R^3$, and let S_3 be the act on V by permuting a basis. Describe* $\mathrm{End}_G(V)$ *(i.e., the set of all G-equivariant maps from V to V).*

Definition 1.4. If $\rho : G \to \mathrm{Aut}_R(V)$ is a representation, a submodule $W \leqslant V$ is *G-stable* if $\rho(g)(W) \subseteq W$ for all $g \in G$.

Exercise 3. *For $G = S_n$ acting by permuting a basis of R^n, as above, find a nontrivial G-stable subspace of R^n.*

Exercise 4. *For $G = (K, +)$ acting on K^2, as in Example 1.2(7), find a nontrivial G-stable subspace.*

Exercise 5 (If you know what a category is!)**.** *Fix a group G and a commutative ring R. Show that the collection of R-linear representations of G and G-equivariant maps between them forms a category.*

2. Group Algebras and their Relationship to Representation Theory

This section is a rehash of its twin in the Artin-Wedderburn worksheet along with a new exercise.

Definition 2.1. Let R be a ring and G be any group. The group ring $R[G]$ is defined as

$$R[G] = \bigoplus_{g \in G} R \cdot g = \left\{ \sum_{g \in G} r_g \cdot g \mid r_g = 0 \text{ for all but finitely many } g \in G \right\},$$

with addition and multiplication defined as follows: let $\sum_{g \in G} r_g \cdot g$ and $\sum_{g \in G} s_g \cdot g$ be elements in $R[G]$ and set

$$\sum_{g\in G} r_g \cdot g + \sum_{g\in G} s_g \cdot g := \sum_{g\in G}(r_g + s_g)\cdot g$$
$$\left(\sum_{g\in G} r_g \cdot g\right)\cdot\left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g,h\in G}(r_g s_h)\cdot gh.$$

The following theorem illustrates the bridge between group algebras and representations.

Theorem 2.1. *Let K be a field, V a K-vector space, and G a group. There is a bijection*

$$\left\{ \begin{array}{c} R\text{-linear representations} \\ \text{of } G \text{ on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R[G]\text{-module structures on } V \\ \text{that extend the given action of } K \end{array} \right\}.$$

Moreover, if V and W are R-linear representations of G, then $\psi : V \to W$ is G-equivariant if and only if it is $R[G]$-linear.

Exercise 6. *Let R be a ring and G be a group. Consider $R[G]$ as a left $R[G]$-module in the usual way; describe the corresponding R-linear representation of G on $R[G]$. We call this the standard R-linear or regular R-linear representation of G.*

3. Irreducible Representations

In this section we will define and play around with irreducible representations of a group G over a ring R. Informally, irreducible representations are the "smallest" types of representations. More formally:

Definition 3.1. We say that an R-linear representation of group G, (M, ρ), is *irreducible* if it has no nontrivial G-stable subspaces.

Exercise 7. *Determine whether or not the following are irreducible representations.*

1. *Let G be any group and R any commutative ring. Consider the trivial representation of G on R^n, with $n \geqslant 2$.*
2. *Let G be a group and $R = \mathbb{C}$. Consider (V, ρ), where $\dim_{\mathbb{C}}(V) = 1$.*
3. *Let $R = \mathbb{C}$. Consider the sign representation of S^n on $\mathbb{C}^n$.*
4. *Let $R = \mathbb{C}$. Consider the sign representation of S^m on $\mathbb{C}^n$ with $m \neq n$.*
5. *Let $R = \mathbb{C}$. Let V be a $\mathbb{C}$-vector space with basis $\{e_1, \ldots, e_n\}$. Consider the following representation of S_n on V defined by $\sigma \cdot e_i := e_{\sigma(i)}$.*
6. *Let $G = D_{2n}$, symmetries of the equilateral polygon on n vertices. Then G acts linearly on $V = \mathbb{R}^2$ by rotations and reflections. Consider the $\mathbb{R}$-linear representation of G on V.*
7. *Let $R = \mathbb{C}$ be a field, $V = \mathbb{C}^2$, and let $G = (\mathbb{C}, +)$. The assignment*
$$\rho : G \to \mathrm{GL}_2(\mathbb{C}) \quad \rho(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$
is a representation. Is it irreducible?

Remark 3.1. Irreducible representations are the "simple modules" of the representation theory world. This statement is actually not too far off from the truth. Indeed:

Exercise 8. *Let R be a commutative ring and G a group. Show that a R-linear representation of G is irreducible if and only if it is a simple $R[G]$-module.*

Exercise 9 (Maschke's Theorem)**.** *Let K be a field and G a finite group such that the characteristic of K does not divide $|G|$. We show that $K[G]$ is left semi-simple.*

1. *More generally, let R be any ring. Prove that R is left semi-simple if and only if every injection $\iota : M \to M'$ of left R-modules splits.*
2. *Let $R = K[G]$ as above and let $\iota : M \to M'$ be an injection of left R-modules. Prove that there is a splitting $j : M' \to M$ as K-vector spaces.*
3. *With the K-linear splitting $j : M' \to M$ you found above, define the map*

$$\rho : M' \to M \qquad \rho(m') = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot \iota(g \cdot m').$$

   Prove that ρ is a $K[G]$ splitting of ι.
4. *Conclude that $K[G]$ is left semi-simple.*

Exercise 10. *Let K be a field and G a finite group such that the characteristic of K does not divide $|G|$. Why does Maschke's Theorem imply every K-linear representation of G decomposes as a direct sum of irreducible representations?*

Exercise 11. *Show that Mashke's Theorem need not hold when the characteristic of K divides $|G|$. Hint: Find a $\mathbb{F}_2$-linear representation of some finite group G (of order divisible by two) that does not decompose as a direct sum of irreducible representations.*

4. From Old Representations to New Representations

In this short section we will sketch out a few constructions of representations using old representations as well as a few tools from our commutative algebra tool box.

Exercise 12. *Let R be a ring and G a group. Suppose M and N are R-linear representations of G.*

1. *How should G act on $\mathrm{Hom}_R(M, N)$ so that the following diagram commutes for all $\phi \in \mathrm{Hom}_R(M, N)$ and for all $g \in G$:*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g}\ ? \\ M & \xrightarrow{g \cdot \phi} & N \end{array}$$

2. *Why is the action you found in the part above a desirable thing to have?*

Definition 4.1. Let R be a ring and G a group. Suppose M and N are R-linear representations of G.

1. We make $\mathrm{Hom}_R(M, N)$ a R-linear representation of G by the action:

$$(g \cdot \phi)(m) = g\phi(g^{-1}m)$$

2. We make $M \otimes_R N$ into a R-linear representation of G by the action:

$$g(m \otimes n) = g(m) \otimes g(n).$$

Exercise 13. *Let K be a field and V a k-linear G-representation for some group G. Let $V^* = \mathrm{Hom}_K(V, K)$ be the K-dual of V. Make V^* into a G-representation following the definition above.*

(1) *Are V and V^* isomorphic as G-representations?*
(2) *If V is irreducible, is V^* also irreducible?*

Exercise 14. *Let L/K be a field extensions and V a K-linear G-representation for some group G. Let $V_L := V \otimes_K L$ be an L-linear G-representation given by the G-action $g \cdot (v \otimes l) = gv \otimes l$. If V is irreducible, is V_L also irreducible?*

Exercise 15. *Let R be a ring and G a group. Let M be a R-linear G-representation.*

(1) *If M and N are irreducible R-linear G-representations, must $V \otimes W$ be irreducible as a G-representation?*
(2) *Let M be an irreducible. Let H be another group and let N be an irreducible R-linear H-representation. Consider the $G \times H$ action on $M \otimes_R N$ given by $(g, h) \cdot (m \otimes n) = gm \otimes hn$. Under this action, is $M \otimes_R N$ an irreducible R-lienar $G \times H$- representation?*

Exercise 16. *Let (M, ρ) and (R, ρ') be R-linear G-representations. Show that there is an G-equivariant isomorphism between $\mathrm{Hom}_R(R, M)$ and M as R-linear G-representations.*