EXERCISES 09/03/2024

These exercises are of varying difficulty. If your group is stuck on a problem, I suggest trying the others first and then go back. I don't expect every group to finish this in our meeting, so if you like, you may work on these during your own time. These are not meant to be turned it; they are just for fun! **Exercises with a * are used later in the text.**

(1) Exercise 4 pg. 2: Show that the natural representation of SL_2 (and GL_2) on K^2 has two orbits. Furthermore show that the stabilizer of $e_1 = (1, 0)$ is $U := \left\{\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \mid s \in K\right\}$, usually dubbed the subgroup of *upper triangular unipotent matrices*, and the stabilizer of any other point $(x, y) \neq (0, 0)$ is conjugate to U.

(2) Exercise 5 pg. 3: Consider the linear action of GL_2 on $K[x, y]$ induced by the natural representation of GL_2 on K^2.
   (a) What is the image of x and y under $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K)$?
   (b) Show that $K[x, y]^{\mathrm{GL}_2(K)} = K[x, y]^{\mathrm{SL}_2} = K$.
   (c) Show that $K[x, y]^U = K[y]$, where $U := \left\{\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \mid s \in K\right\}$.

(3) Exercise 6 pg. 4: Determine the invariant rings $K[M_2(K)]^U$ and $K[M_2(K)]^T$ under left multiplication by the subgroup U (as above) and $T := \left\{\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mid t \in K^\times\right\}$

(4) Exercise 8* pg. 5: Consider the following symmetric functions $n_j = x_1^j + x_2^j + \ldots + x_n^j \in K[x_1, \ldots, x_n]$ called the *power sums* or *Newton functions*.
   (a) Prove the following formulas due to Newton:

   $$(-1)^{j+1} j\sigma_j = n_j - \sigma_1 n_{j-1} + \sigma_2 n_{j-2} - \cdots + (-1)^{j-1}\sigma_{j-1} n_1$$

   for all $j = 1, \ldots, n$. Here, $\sigma_k = \sum_{j_1 < j_2 < \ldots < j_k} x_{j_1} x_{j_2} \cdots x_{j_k}$. **There is a hint in the text.**
   (b) Show that in characteristic 0 the power sums $n_1, \ldots, n_n$ generate the symmetric functions.

(5) Exercise 12* pg. 6: Let V be a K-vector space, not necessarily finite dimensional, let L/K be a field extension and $U_i \subseteq V$, where i is in some indexing set I, is a family of subspaces. Then

$$L \otimes_K \left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} (L \otimes_K U_i) \subset L \otimes_K V.$$

(6) Exercise 16* pg. 8: Let $H \subset G \subset GL(W)$ be subgroups and assume that H is Zariski-dense in G. Then a linear subspace $U \subset W$ is H-stable if and only if it is G-stable. Moreover, we have $K[W]^H = K[W]^G$.