EXERCISES 09/10/2024

These exercises are of varying difficulty. If your group is stuck on a problem, I suggest trying the others first and then go back. I don't expect every group to finish this in our meeting, so if you like, you may work on these during your own time. These are not meant to be turned it; they are just for fun! **Exercises with a * are used later in the text.**

Please feel free to work on past problems, as well!

(1) Exercise 14 pg. 7: Let $k \subset K$ be an infinite subfield. Prove that $\mathrm{GL}_n(k)$ is Zariski dense in $\mathrm{GL}_n(K)$ and that $\mathrm{SL}_n(k)$ is Zariski-dense in $\mathrm{SL}_n(K)$. **Hint:** Proof that k^n is Zariski-dense in K^n.

(2) Exercise 18 pg. 8: Show that the map $\phi \mapsto \phi^*$ defines a bijection between the set of morphisms $W \to V$ and the set of algebra homomorphisms $K[V] \to K[W]$. In addition, prove the following:
   (a) If $\psi : V \to U$ is a another morphism, then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
   (b) ϕ^* is injective if and only if the image $\phi(W)$ is Zariski-dense in V.
   (c) If ϕ^* is surjective, then ϕ is injective.

(3) Exercise 19 pg. 8: Let $\phi : V \to W$ be a morphism of vector spaces and let $X \subset Y \subset V$ be subsets, where X is Zariski-dense in Y. Prove that $\phi(X)$ is Zariski-dense in $\phi(Y)$.

(4) Let V be a K-vector space of dimension n. Assume $\mathrm{char}(K) \neq 2$. Prove that $\mathrm{Sym}^2(V) \oplus \bigwedge^2 V \cong V \otimes_k V$. **Hint:** If $\bigwedge^2 V$ is foreign to you, find someone who knows about it and ask them to show you what it is.

(5) Exercise 21 pg. 9: Denote by $K[\mathrm{SL}_2(K)]$ the algebra of functions $f|_{\mathrm{SL}_2(K)}$ (f restricted to $\mathrm{SL}_2(K)$) where f is a polynomial function on $M_2(K)$. Show that the kernel of the restriction map $\mathrm{res} : K[M_2(K)] \to K[\mathrm{SL}_2(K)]$ is the ideal generated by 1 minus the determinant; we call the determinant det. **Hints**: after choosing a basis from $M_2(K)^*$ what does $\det \in K[M_2(K)]$ look like? First prove the result when K is algebraically closed. The general case follows by showing that $SL_2(K)$ is Zariski-dense in $SL_2(\overline{K})$.

(6) Exercise 29 pg. 12: Let L/K be a field extension. For any K-vector space V, we set $V_L := V \otimes_K L$. If G is a group and V is a G-module, then V_L is also a G-module (what is the G-action on V_L?) Prove the following
   (a) Show that $V_L^G = (V^G)_L$.
   (b) Prove that $L[V]^G = L \otimes_K K[V]^G$.
   (c) If $U \subset V$ is a G-submodule and if U_L has a G-stable complement in V_L, then U has a G-stable complement in V. **Hint**: Consider the natural map $\mathrm{Hom}(V, U) \to \mathrm{Hom}(U, U)$ and use that $\mathrm{Hom}(V_L, W_L) \cong \mathrm{Hom}(V, W)_L$.

(d) We say that a representation of G on W is **completely reducible** if W is a direct sum of irreducible representations of G. A representation of G on W is said to be **irreducible** if it has no proper nonzero G-stable subspace. Show that if the representation of G on V_L is completely reducible, then so is the representation of G on V.

(7) Exercise 30* pg. 13: Let A be a commutative algebra and let G be a group of algebra automorphism of A. Assume that the representation of G on A is completely reducible. Show that the subalgebra A^G of invariant has a canonical G-stable complement and the corresponding G-equivariant projection $p : A \to A^G$ satisfies the relation $p(hf) = hp(f)$ for all $h \in A^G$ and $f \in A$.